Kuwait University | Department of Mathematics

Math 102 : Calculus II	Summer Semester 2010	In-Term Test 1

Saturday 10 July 2010

Duration: 90 minutes | Total marks: 25

Justify all your answers

1. State the definition that a function f with domain A is one-to-one on A. [1 mark]

2. Let $f(x) = e^{-x} - x$ for $x \in \mathbb{R}$.

   (a) Show that f is one-to-one on $\mathbb{R}$. [1 mark]

   (b) Determine the domain of f^{-1}. [1 mark]

   (c) Explain why the point $(1, 0)$ is on the graph of f^{-1}, and find the slope of the tangent line at this point. [2 marks]

3. Use the definition of the natural logarithm function to prove that

$$\ln(ab) = \ln a + \ln b$$

for any positive numbers a and b. [2 marks]

4. Use logarithmic differentiation to find $f'(0)$ when

$$f(x) = \frac{(x+1)^x\,(2\arcsin x - 3)\,e^x}{\sqrt{x^2+1}}.$$ [3 marks]

5. Solve the equation

$$e^{3x} + \sinh x = 0.$$ [2 marks]

6. Evaluate the following.

   (a) $\displaystyle\int \frac{10^x}{100^x + 81}\,dx.$ [3 marks]

   (b) $\displaystyle\int \frac{t + \sqrt{9-t^2}}{9-t^2}\,dt.$ [3 marks]

   (c) $\displaystyle\int \frac{\tanh x}{\sqrt{3\sinh^2 x + \cosh^2 x - 1}}\,dx.$ [3 marks]

7. Evaluate the following.

   (a) $\displaystyle\lim_{x\to 0}\left(\frac{e^x}{\sin x} - \frac{1}{x}\right).$ [2 marks]

   (b) $\displaystyle\lim_{x\to 1}(1 + \ln x)^{1/(x-1)}.$ [2 marks]

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SOLUTION

1. f is one-to-one on A if $f(x_1) \neq f(x_2)$ whenever $x_1 \in A$, $x_2 \in A$, and $x_1 \neq x_2$.

2. (a) $$f'(x) = -e^{-x} - 1 < 0 \quad \text{for all } x \in \mathbb{R}.$$
   So f is decreasing on $\mathbb{R}$. This means that $f(x_1) > f(x_2)$ whenever $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$, and $x_1 < x_2$.

   (b) The domain of f^{-1} is the range of f. Since $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and f is continuous on $\mathbb{R}$, the range of f is $\mathbb{R}$. Answer: $\mathbb{R}$.

   (c) The point $(1, 0)$ is on the graph of f^{-1} if and only if the point $(0, 1)$ is on the graph of f. The point $(0, 1)$ is on the graph of f because $f(0) = e^{-0} - 0 = 1$.

   The slope of the tangent line is
   $$\left(f^{-1}\right)'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-e^{-0} - 1} = -\frac{1}{2}.$$

3. By the definition of the natural logarithm function,
$$\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt = \ln a + \int_a^{ab} \frac{1}{t}\,dt.$$
By the substitution $t = au$, which implies that $dt = a\,du$,
$$\int_a^{ab} \frac{1}{t}\,dt = \int_1^{b} \frac{1}{au}\,a\,du = \int_1^{b} \frac{1}{u}\,du = \ln b.$$
Thus $\ln(ab) = \ln a + \ln b$.

4. Taking the natural logarithm of the absolute value[1] and using the laws of logarithms,
$$\ln|f(x)| = \pi \ln(x+1) + \ln|2\arcsin x - 3| + x - \tfrac{1}{2}\ln(x^2+1).$$
Differentiating,
$$\frac{f'(x)}{f(x)} = \pi\frac{1}{x+1} + \frac{1}{2\arcsin x - 3}\,\frac{2}{\sqrt{1-x^2}} + 1 - \tfrac{1}{2}\,\frac{1}{x^2+1}\,2x.$$
So,
$$f'(0) = f(0)\left(\pi + \tfrac{2}{-3} + 1 - 0\right) = f(0)\left(\pi + \tfrac{1}{3}\right) = -3\left(\pi + \tfrac{1}{3}\right) = -3\pi - 1.$$

5. Using the definition of the hyperbolic sine function the equation is
$$e^{3x} + \frac{e^x - e^{-x}}{2} = 0.$$
Multiplying by $2e^x$, it becomes
$$2e^{4x} + e^{2x} - 1 = (e^{2x}+1)(2e^{2x}-1) = 0.$$
Therefore,
$$2e^{2x} - 1 = 0 \implies e^{2x} = \tfrac{1}{2} \implies 2x = \ln\tfrac{1}{2} = -\ln 2 \implies x = -\tfrac{1}{2}\ln 2.$$

6. (a) By the substitution $u = 10^x$, which implies that $du = 10^x(\ln 10)\,dx$,
$$\begin{aligned}\int \frac{10^x}{100^x + 81}\,dx &= \int \frac{1}{u^2+81}\,\frac{du}{\ln 10} = \frac{1}{\ln 10}\int \frac{1}{u^2+9^2}\,du \\ &= \frac{1}{\ln 10}\,\frac{1}{9}\arctan\!\left(\frac{u}{9}\right) + C = \frac{1}{9\ln 10}\arctan\!\left(\frac{10^x}{9}\right) + C.\end{aligned}$$

[1]It is essential that the absolute value be taken, because $f(0) < 0$.

(b)

$$\int \frac{t+\sqrt{9-t^2}}{9-t^2}\,dt = \int \frac{t}{9-t^2}\,dt + \int \frac{1}{\sqrt{9-t^2}}\,dt.$$

Substituting $u = 9 - t^2$ in the first integral on the right-hand side, and substituting $t = 3v$ in the last integral, which implies that $du = -2t\,dt$ and $dt = 3\,dv$ respectively,

$$\begin{aligned}\int \frac{t+\sqrt{9-t^2}}{9-t^2}\,dt &= \int \frac{1}{u}\frac{du}{-2} + \int \frac{1}{\sqrt{9-(3v)^2}}\,3\,dv \\ &= -\tfrac{1}{2}\int \frac{1}{u}\,du + \int \frac{1}{\sqrt{1-v^2}}\,dv = -\tfrac{1}{2}\ln u + \arcsin v + C \\ &= -\tfrac{1}{2}\ln(9-t^2) + \arcsin(\tfrac{1}{3}t) + C.\end{aligned}$$

(c) By[2] the substitution $u = \operatorname{sech} x$, which implies that $\cosh^2 x = u^{-2}$, $\sinh^2 x = \cosh^2 x - 1 = u^{-2} - 1$, and $du = -\operatorname{sech} x \tanh x\,dx$,

$$\begin{aligned}\int \frac{\tanh x}{\sqrt{3\sinh^2 x + \cosh^2 x - 1}}\,dx &= \int \frac{1}{\sqrt{3(u^{-2}-1)+u^{-2}-1}}\,\frac{du}{-u} = \cdots \\ &= \tfrac{1}{2}\int \frac{-1}{\sqrt{1-u^2}}\,du = \tfrac{1}{2}\arccos u + C \\ &= \tfrac{1}{2}\arccos(\operatorname{sech} x) + C.\end{aligned}$$

7. (a)

$$\lim_{x\to 0}\left(\frac{e^x}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{xe^x - \sin x}{x\sin x}.$$

This limit is indeterminate of the type $0/0$.

$$\lim_{x\to 0}\frac{\frac{d}{dx}(xe^x - \sin x)}{\frac{d}{dx}(x\sin x)} = \lim_{x\to 0}\frac{e^x + xe^x - \cos x}{\sin x + x\cos x}.$$

This is also indeterminate of type $0/0$.

$$\lim_{x\to 0}\frac{\frac{d}{dx}(e^x + xe^x - \cos x)}{\frac{d}{dx}(\sin x + x\cos x)} = \lim_{x\to 0}\frac{2e^x + xe^x + \sin x}{2\cos x - x\sin x} = \frac{2}{2} = 1.$$

By l'Hospital's Rule, the latter implies that

$$\lim_{x\to 0}\frac{e^x + xe^x - \cos x}{\sin x + x\cos x} = 1.$$

By a further application of l'Hospital's Rule, the above implies that

$$\lim_{x\to 0}\frac{xe^x - \sin x}{x\sin x} = 1.$$

(b) Let

$$y = (1+\ln x)^{1/(x-1)}.$$

Then

$$\lim_{x\to 1}\ln y = \lim_{x\to 1}\frac{\ln(1+\ln x)}{x-1}.$$

This limit is indeterminate of the type $0/0$.

$$\lim_{x\to 1}\frac{\frac{d}{dx}\ln(1+\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{\frac{1}{1+\ln x}\,\frac{1}{x}}{1} = 1.$$

So, by l'Hospital's Rule, $\lim_{x\to 1}\ln y = 1$. Hence, $\lim_{x\to 1} y = e^1 = e$.

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[2]Alternative answers are $-\frac{1}{2}\arcsin(\operatorname{sech} x) + C$, $\frac{1}{2}\sec^{-1}(\cosh x) + C$, $-\frac{1}{2}\csc^{-1}(\cosh x) + C$, $\frac{1}{2}\arcsin|\tanh x| + C$, $-\frac{1}{2}\arccos|\tanh x| + C$, $\frac{1}{2}\arctan|\sinh x| + C$, $-\frac{1}{2}\cot^{-1}|\sinh x| + C$, $\frac{1}{2}\cot^{-1}|\operatorname{csch} x| + C$, $-\frac{1}{2}\arctan|\operatorname{csch} x| + C$, $\frac{1}{2}\csc^{-1}|\coth x| + C$, $-\frac{1}{2}\sec^{-1}|\coth x| + C$, $\arctan(e^{|x|}) + C$, $-\cot^{-1}(e^{|x|}) + C$, $\cot^{-1}(e^{-|x|}) + C$, and $-\arctan(e^{-|x|}) + C$.